

Bounding regions to plane steepest descent curves of quasi convex families *

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Abstract

Two dimensional steepest descent curves (SDC) for a quasi convex family are considered; the problem of their extensions (with constraints) outside of a convex body K is studied. It is shown that possible extensions are constrained to lie inside of suitable bounding regions depending on K . These regions are bounded by arcs of involutes of ∂K and satisfy many inclusions properties. The involutes of the boundary of an arbitrary plane convex body are defined and written by their support function. Extensions SDC of minimal length are constructed. Self contracting sets (with opposite orientation) are considered, necessary and/or sufficient conditions for them to be subsets of a SDC are proved.

1 Introduction

Let u be a smooth function defined in a convex body $\Omega \subset \mathbb{R}^n$. Let $Du(x) \neq 0$ in $\{x \in \Omega : u(x) > \min u\}$. A classical steepest descent curve of u is a rectifiable curve $s \rightarrow x(s)$ solution to

$$\frac{dx}{ds} = \frac{Du}{|Du|}(x(s)).$$

Classical steepest descent curves are the integral curves of a unit field normal to the sublevel sets of the given function u . We are interested in steepest descent curves that are integral curves to a unit field normal to the family $\{\Omega_t\} := \{x : u(x) \leq t\}$ of the sublevel sets for a quasi convex function u (see

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Definition 2.4); $\{\Omega_t\}$ will be called a quasi convex family as in [6]. Sharp bounds about the length of the steepest descent curves for a quasi convex family, have been proved in [8],[11],[12]. The geometry of these curves, equivalent definitions, related questions and generalizations have been studied in [1], [3], [4], [9], [10].

In the above works, it has been proved that *steepest descent curves for a quasi convex family* (SDC) can be characterized as bounded oriented rectifiable curves $\gamma \subset \mathbb{R}^n$, with a locally lipschitz continuous parameterization $T \ni t \rightarrow x(t)$ satisfying

$$\langle \dot{x}(t), x(\tau) - x(t) \rangle \leq 0, \quad \text{a.e. } t \in T, \quad \forall \tau \leq t; \quad (1)$$

$\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^n . Let an ordering \preceq be chosen on γ , according to the orientation; let us denote

$$\gamma_x = \{y \in \gamma : y \preceq x\}. \quad (2)$$

The curves γ satisfying (1) are SDC for the related quasi convex family $\Omega_t := \text{co}(\gamma_{x(t)})$, where $\text{co}(A)$ denotes the convex hull of the set A .

The SDC could also be characterized in an equivalent way as *self-distancing curves*, namely oriented (\preceq) continuous curves with the property that the distance of x to an arbitrarily fixed previous point x_1 is not decreasing:

$$\forall x_1, x_2, x_3 \in \gamma, x_1 \preceq x_2 \preceq x_3 \Rightarrow |x_2 - x_1| \leq |x_3 - x_1|. \quad (3)$$

In [9] self-distancing curves are called self-expanding curves. With the opposite orientation these curves have been also introduced, studied and called self-approaching curves (see [8]), or self-contracting curves (see [4]).

An important property that will be used later is the property of distancing from a set A :

Definition 1.1. *Given a set A , an absolutely continuous curve $\gamma, T \ni t \rightarrow x(t)$ has the distancing from A property if it satisfies*

$$\langle \dot{x}(t), y - x(t) \rangle \leq 0, \quad \text{a.e. } t \in T, \quad \text{and} \quad \forall y \in A. \quad (4)$$

Steepest descent curves (or self-distancing curves) γ that also satisfy the above property with respect to a convex set K will be called SDC_K . Of course if γ is a SDC and $x \in \gamma$ then $\gamma \setminus \gamma_x$ is a $SDC_{\text{co}(\gamma_x)}$.

In the present work we are interested on the behaviour and properties of a **plane** SDC γ beyond its final point x_0 . The principal goal of the paper is to show that conditions (1) or (3) imply constraints for possible extensions of the curve γ beyond x_0 ; these constraints are written as bounding regions for the possible extensions of γ_{x_0} .

Let us outline the content of our work. In §2 introductory definitions are given and covering maps for the boundary of a plane convex set, needed

for later use, are introduced. In §3 the involutes of the boundary of a plane convex body are introduced and some of their properties are proved.

In §4 plane regions depending on the convex body $co(\gamma_{x_0})$ have been defined; these regions fence in or fence out the possible extensions of γ_{x_0} . The boundary of these sets consists of arcs of involutes of convex bodies, constructed in §3. As an application, in §4.1 the following problem has been studied: given a convex set K , $x_0 \in \partial K$, $x_1 \notin K$ is it possible to construct a SDC_K joining x_0 to x_1 ? Minimal properties of this connection have been introduced and studied. In §5 sets of points more general than SDC are studied. A set $\sigma \subset \mathbb{R}^2$ (not necessarily a curve) of ordered points satisfying (3) will be called *self-distancing set*, see also Definition 2.1; with the opposite order, σ was called self-contracting in [3] and many properties of these sets, as only subsets of self contracting curves, were there obtained. A natural question arises: does it exist a steepest descent curve $\gamma \supset \sigma$? Examples, necessary and/or sufficient conditions are given when σ consists of a finite or countable number of points $x_i \in \mathbb{R}^2$ and/or steepest descent curves $\gamma^i \subset \mathbb{R}^2$.

In the present work the two dimensional case is studied. Similar results for the n dimensional case are an open problem stated at the end of the work.

2 Preliminaries and definitions

Let

$$B(z, \rho) = \{x \in \mathbb{R}^n : |x - z| < \rho\}, \quad S^{n-1} = \partial B(0, 1) \quad n \geq 2.$$

A not empty, compact convex set K of \mathbb{R}^n will be called a *convex body*. From now on, K will **always be a convex body not reduced to a point**. $Int(K)$ and ∂K denote the interior of K and the boundary of K , $|\partial K|$ denotes its length, $cl(K)$ is the closure of K , $Aff(K)$ will be the smallest affine space containing K ; $relint K$ and $\partial_{rel} K$ are the corresponding subsets in the topology of $Aff(K)$. For every set $S \subset \mathbb{R}^n$, $co(S)$ is the convex hull of S .

Let $q \in K$; the *normal cone* at q to K is the closed convex cone

$$N_K(q) = \{x \in \mathbb{R}^n : \langle x, y - q \rangle \leq 0 \quad \forall y \in K\}. \quad (5)$$

When $q \in Int(K)$, then $N_K(q)$ reduces to zero.

The *tangent cone*, or support cone, of K at a point $q \in \partial K$ is given by

$$T_K(q) = cl \left(\bigcup_{y \in K} \{s(y - q) : s \geq 0\} \right).$$

In two dimensions cones will be called sectors.

Let K be a convex body and p be a point. A simple cap body K^p is:

$$K^p = \bigcup_{0 \leq \lambda \leq 1} \{\lambda K + (1 - \lambda)p\} = \text{co}(K \cup \{p\}). \quad (6)$$

Cap bodies properties can be found in [2],[14].

2.1 Self-distancing sets and steepest descent curves

Let us recall the following definitions

Definition 2.1. *Let us call **self-distancing set** a bounded subset σ of \mathbb{R}^n , linearly ordered (by \preceq), with the property:*

$$x_1, x_2, x_3 \in \sigma, \text{ and } x_1 \preceq x_2 \preceq x_3 \implies |x_2 - x_1| \leq |x_3 - x_1|. \quad (7)$$

The self-distancing sets has been introduced in [3] with the opposite order. If a self-distancing set σ is a closed connected set, not reduced to a point, then it can be proved that σ is a steepest descent curve (see [3, Theorem 3.3], [9, Theorem 4.10]) and it will also be called a **self-distancing curve** γ .

The short name SDC will be used for self-distancing curves (steepest descent curves) in all the paper.

Definition 2.2. *Let K be a convex body, $\gamma \subset \mathbb{R}^2 \setminus \text{relint } K$ will be called a **self-distancing curve from K** (denoted SDC_K) if:*

- (i) γ is a self-distancing curve,
- (ii) $\gamma \cap \partial_{\text{rel}} K \neq \emptyset$,
- iii) γ has the **distancing from K** property:

$$\forall y \in K, \forall x, x_1 \in \gamma : x \preceq x_1 \implies |x - y| \leq |x_1 - y|. \quad (8)$$

When (ii) does not hold, that is $\gamma \cap \partial_{\text{rel}} K = \emptyset$, γ will be called a **deleted self-distancing curve from K** .

Remark 2.3. *Let γ be a SDC_K , then γ has an absolutely continuous parameterization, thus property (8) for γ is equivalent to (4).*

Nested families of convex sets have been introduced and studied by De Finetti [5] and Fenchel [6]. Let us recall some definitions.

Definition 2.4. *Let T be a real interval. A **convex stratification** (see [5]) is a not empty family \mathfrak{K} of convex bodies $\Omega_t \subset \mathbb{R}^n$, $t \in T \subset \mathbb{R}$, linearly strictly ordered by inclusion ($\Omega_1 \subset \Omega_2$, $\Omega_1 \neq \Omega_2$), with a maximum set ($\max \mathfrak{K}$) and a minimum set ($\min \mathfrak{K}$).*

Let $\mathfrak{K} = \{\Omega_t\}_{t \in T}$ be a convex stratification. If for every $s \in T \setminus \{\max T\}$ the property:

$$\bigcap_{t > s} \Omega_t = \Omega_s$$

holds, then as in [6], $\mathfrak{K} = \{\Omega_t\}_{t \in T}$ will be called a **quasi convex family**.

An important quasi convex family associated to a continuous self-distancing curve from K , $\gamma: t \rightarrow x(t)$ is $\mathfrak{K} = \{\Omega_t\}_{t \in T}$, where

$$\Omega_t = \text{co}(\gamma_{x(t)} \cup K).$$

The couple (γ, \mathfrak{K}) is special case of *Expanding Couple*, a class introduced in [9].

Remark 2.5. If $\gamma \in \text{SDC}_K$, then for all $x \in \gamma$ the curve $(\gamma \setminus \gamma_x) \cup \{x\}$ is a self-distancing curve from the convex hull of the set $\gamma_x \cup K$.

This fact is a direct consequence of the following

Proposition 2.6 ([9], Lemma 4.9). Let $p, q, y_i \in \mathbb{R}^n$, $i = 1, \dots, s$. If

$$|p - y| \leq |q - y|, \quad \text{for } y = y_i, i = 1, \dots, s \quad (9)$$

then the same holds for every $y \in \text{co}(\{y_i, i = 1, \dots, s\})$. In (9) inequality \leq can be changed with strict inequality.

2.2 The support function of a plane convex body

Let $K \subset \mathbb{R}^n$ be a convex body not reduced to a point.

For a convex body K , the *support function* is defined as

$$H_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . For $n = 2$, $\vartheta \in \mathbb{R}$, let $\theta = (\cos \vartheta, \sin \vartheta) \in S^1$ and $h_K(\vartheta) := H_K(\theta)$, it will be denoted $h(\vartheta)$ if no ambiguity arises.

For every $\theta \in S^1$ there exists at least one point $x \in \partial K$ such that:

$$\langle \theta, y - x \rangle \leq 0 \quad \forall y \in K; \quad (10)$$

this means that the line through x orthogonal to θ supports K . For every $x \in \partial K$ let \widehat{N}_x the set of $\theta \in S^1$ such that (10) holds. Let $F(\theta)$ be the set of all $x \in \partial K$ satisfying (10). If ∂K is strictly convex at the direction θ then $F(\theta)$ reduces to one point and it will be denoted by $x(\theta)$.

Definition 2.7. *The set valued map: $G : \partial K \rightarrow S^1, \partial K \ni x \rightarrow \widehat{N}_x \subset S^1$, is the generalized Gauss map; $x \in \partial K$ is a vertex on ∂K iff \widehat{N}_x is a sector with interior points. The set valued map $F : S^1 \rightarrow \partial K, S^1 \ni \theta \rightarrow F(\theta) \subset \partial K$ is the reverse generalized Gauss map; $F(\theta)$ is a closed segment, possibly reduced to a single point, and it will be called 1-face when it has interior points.*

Let P be the covering map

$$P : \mathbb{R} \rightarrow S^1, \mathbb{R} \ni \vartheta \rightarrow \theta = (\cos \vartheta, \sin \vartheta) \in S^1.$$

Let $L = |\partial K|$, let $s \rightarrow x_l(s), 0 \leq s < L$ ($s \rightarrow x_r(s), 0 \leq s < L$) be the parametric representations of ∂K depending on the arc length counter-clockwise (clockwise) with an initial point (not necessarily the same). Let us extend $x_l(\cdot)$ and $x_r(\cdot)$ by defining

$$x_l(s) := x_l(s - kL) \quad \text{if} \quad kL \leq s < (k+1)L, \quad (k \in \mathbb{Z}).$$

Similarly for x_r .

Let us fix $x_0 \in \partial K$, $\theta_0 \in G(x_0)$, $\theta_0 = (\cos \vartheta_0, \sin \vartheta_0)$, $\vartheta_0 \in \mathbb{R}$.

For later use, we need to have $x_0 = x_l(s_0) = x_r(s_0)$; this can be realized by choosing suitable initial points for the parameterizations x_l and x_r .

Then

$$x_l(s_0 + s) = x_r(s_0 + L - s), \quad \forall s \in \mathbb{R}.$$

The maps

$$x_l : \mathbb{R} \rightarrow \partial K, \quad x_r : \mathbb{R} \rightarrow \partial K,$$

are covering maps.

The initial parameters will be

$$x_0 = x_l(s_0) = x_r(s_0) \in \partial K, S^1 \ni \theta_0 \in \overset{-1}{F}(x_0), \mathbb{R} \ni \vartheta_0 \in \overset{-1}{P}(\theta_0),$$

($\overset{-1}{F}(x_0), \overset{-1}{P}(\theta_0)$ are the back images of F, P respectively). Let $k \in \mathbb{Z}$. Let us define, for $\vartheta_0 + 2k\pi < \vartheta < \vartheta_0 + 2(k+1)\pi$:

$$s_{l+}(\vartheta) := \sup\{s \in \mathbb{R} : kL < s \leq (k+1)L, x_l(s) \in F(P(\vartheta))\}; \quad (11)$$

if $\vartheta = \vartheta_0 + 2k\pi$

$$s_{l+}(\vartheta) := \sup\{s \in \mathbb{R} : kL \leq s < (k+1)L, x_l(s) \in F(P(\vartheta))\}. \quad (12)$$

Similarly, let us define for $\vartheta_0 + 2(k-1)\pi < \vartheta < \vartheta_0 + 2k\pi$:

$$s_{r-}(\vartheta) := \inf\{s \in \mathbb{R} : kL \leq s < (k+1)L, x_r(s) \in F(P(\vartheta))\}; \quad (13)$$

if $\vartheta = \vartheta_0 + 2k\pi$

$$s_{r-}(\vartheta) := \inf\{s \in \mathbb{R} : (k-1)L < s \leq kL, x_r(s) \in F(P(\vartheta))\}. \quad (14)$$

The function s_{l+} is increasing in \mathbb{R} , right continuous and with left limits (so called *cadlag* function). Similar properties hold for $-s_{r-}$. Let us recall that a *cadlag* increasing function $s(\vartheta)$, $\vartheta \in \mathbb{R}$ has a right continuous inverse defined as

$$\vartheta(s) = \inf\{\vartheta : s(\vartheta) > s\}.$$

Let $\vartheta_{l+}(\cdot)$ the right continuous inverse of $s_{l+}(\cdot)$. Let $s \rightarrow \vartheta_{r-}(s)$ the opposite of the right continuous inverse of $-s_{r-}(\cdot)$.

Let us introduce for simplicity

$$\mathbf{n}_\vartheta := (\cos \vartheta, \sin \vartheta), \quad \mathbf{t}_\vartheta := (-\sin \vartheta, \cos \vartheta).$$

Let $\vartheta \rightarrow h(\vartheta)$ be the support function of K .

It is well known ([7]) that, if ∂K is C_+^2 (that is $\partial K \in C^2$, with positive curvature), then h is C^2 and the counterclockwise element arc ds of ∂K is given by

$$ds = (h + \ddot{h})d\vartheta. \quad (15)$$

$h(\vartheta) + \ddot{h}(\vartheta)$ is the positive radius of curvature; moreover the reverse Gauss map $F : \theta \rightarrow x \in \partial K$ is a 1-1 map given by

$$x(\theta) := h(\vartheta)\mathbf{n}_\vartheta + \dot{h}(\vartheta)\mathbf{t}_\vartheta, \quad \vartheta \in \bar{P}^{-1}(\theta). \quad (16)$$

The previous formula also holds for an arbitrary convex body, for every ϑ such that $F(\theta)$ is reduced to a point, see [2]. Let us recall that a real valued function $x \rightarrow f(x)$ is called semi convex on \mathbb{R} when there exists a positive constant C such that $f(x) + Cx^2$ is convex on \mathbb{R} . From (15) the function $\vartheta \rightarrow h(\vartheta) + \frac{1}{2}\vartheta^2 \max h$ is convex on \mathbb{R} , thus h is semi convex. In the case that K is an arbitrary convex body, by approximation arguments with C_+^2 convex bodies, see [14], it follows that the support function of K is also semi convex. As consequence h is Lipschitz continuous, it has left (right) derivative \dot{h}_- (respectively \dot{h}_+) at each point, which is left (right) continuous. Moreover at each point the right limit of \dot{h}_- is \dot{h}_+ and the left limit of \dot{h}_+ is \dot{h}_- , see [13, pp. 228].

It is not difficult to show (from (16), with a right limit argument) that for an arbitrary convex body, for $\vartheta \in \mathbb{R}$, the formula

$$x_l(s_{l+}(\vartheta)) = h(\vartheta)\mathbf{n}_\vartheta + \dot{h}_+(\vartheta)\mathbf{t}_\vartheta \quad (17)$$

holds. Similarly the formula

$$x_r(s_{r-}(\vartheta)) = h(\vartheta)\mathbf{n}_\vartheta + \dot{h}_-(\vartheta)\mathbf{t}_\vartheta \quad (18)$$

holds.

If ∂K is not strictly convex at the direction $\theta = (\cos \vartheta, \sin \vartheta)$ then h is not differentiable at ϑ and

$$\dot{h}_+(\vartheta) - \dot{h}_-(\vartheta) = |x_l(s_{l+}(\vartheta)) - x_r(s_{r-}(\vartheta))| = |F(\theta)|. \quad (19)$$

If $x_1, x_2 \in \partial K$ let us define $\text{arc}^+(x_1, x_2)$ the set of points of ∂K between x_1 and x_2 according to the counterclockwise orientation of ∂K , and $\text{arc}^-(x_1, x_2)$ the set of points between x_1 and x_2 , according to the clockwise orientation; $|\text{arc}^\pm(x_1, x_2)|$ denote their length.

Remark 2.8. *It is well known that a sequence of convex body $K^{(n)}$ converges to K if and only if the corresponding sequence of support functions converges in the uniform norm, see [14, pp. 66]. Moreover as the two sequences of the end points of a closed connected arc of $\partial K^{(n)}$ converge, then the sequence of the corresponding arcs converges to a connected arc of ∂K and the sequence of the corresponding lengths converges too.*

Proposition 2.9. *Let K be a convex body and h its support function, then*

$$s_{l+}(\vartheta) - s_{l+}(\vartheta_0) = \int_{\vartheta_0}^{\vartheta} h(\tau) d\tau + \left(\dot{h}_+(\vartheta) - \dot{h}_+(\vartheta_0) \right), \quad \forall \vartheta \geq \vartheta_0; \quad (20)$$

$$s_{r-}(\vartheta_0) - s_{r-}(\vartheta) = \int_{\vartheta}^{\vartheta_0} h(\tau) d\tau + \left(\dot{h}_-(\vartheta) - \dot{h}_-(\vartheta_0) \right), \quad \forall \vartheta \leq \vartheta_0. \quad (21)$$

Proof. For every convex body K not reduced to a point the function $\vartheta \rightarrow s_{l+}(\vartheta)$ is defined everywhere and satisfies the weak form of (15), namely:

$$- \int_{\mathbb{R}} s_{l+}(\eta) \dot{\phi}(\eta) d\eta = \int_{\mathbb{R}} (\phi + \ddot{\phi})(\eta) h(\eta) d\eta, \quad \forall \phi \in C_0^\infty(\mathbb{R}). \quad (22)$$

Using the fact that $\vartheta \rightarrow h(\vartheta)$ is Lipschitz continuous, integrating by parts (22), the formula

$$- \int_{\mathbb{R}} s_{l+}(\eta) \dot{\phi}(\eta) d\eta = - \int_{\mathbb{R}} \dot{\phi}(\eta) \left(\int_0^\eta h(\tau) d\tau + \dot{h}(\eta) \right) d\eta, \quad \forall \phi \in C_0^\infty(\mathbb{R}) \quad (23)$$

holds. Thus

$$s_{l+}(\eta) = c + \int_0^\eta h(\tau) d\tau + \dot{h}(\eta), \quad \text{a.e.}$$

with c constant. Passing to the right limit, the equality

$$s_{l+}(\eta) = c + \int_0^\eta h(\tau) d\tau + \dot{h}_+(\eta), \quad \forall \eta \in \mathbb{R}$$

holds. The formula (20) follows, by computing $s_{l+}(\vartheta) - s_{l+}(\vartheta_0)$, using the previous equality. Similarly (21) is proved. \square

3 Involutives of a closed convex curve

Definition 3.1. *Let I be an interval. A plane curve $I \ni t \rightarrow x(t)$ is convex if at every point x it has right tangent vector $T^+(x)$ and $\arg(T^+(x(t)))$ is a not decreasing function.*

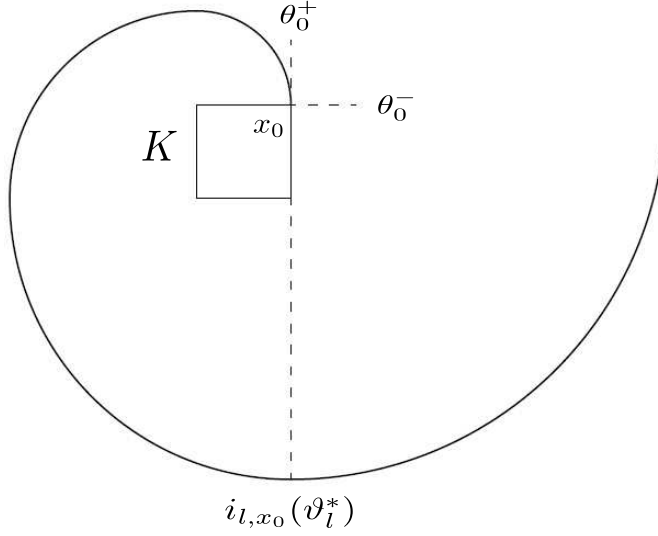


Figure 1: Left involute of a square

Let $s \rightarrow x(s)$ be the arc length parameterization of a smooth curve; the classical definition of involute starting at a point $x_0 = x(s_0)$ of the curve $x(\cdot)$ is

$$i(s) = x(s) - (s - s_0)x'(s) \quad s \geq s_0. \quad (24)$$

Let us notice that s is the arc length of the curve, not of the involute; if $s_0 = 0$, then the starting point of the involute coincides with the starting point of the curve. It is easy to construct an involute of a convex polygonal line (even if the classical definition (24) does not work) by using arcs of circle centered at its corner points; moreover the involute depends on the orientation of the curve.

In this section, involutes for the boundary of an arbitrary plane convex body K , not reduced to a point, will be defined. The assumption that K is an arbitrary convex body is needed to work with the involutes of the convex sets, not smooth, studied in §4.

Let $K \in C_+^2$; let x_0 be a fixed point of ∂K , $s \rightarrow x(s)$ can be the clockwise parameterization of ∂K or the counterclockwise parameterization. Since there exist two orientations, then two different involutes have to be considered. As noted previously one can assume that the parameterizations of ∂K have been chosen so that $x_0 = x_l(s_0) = x_r(s_0)$.

Definition 3.2. Let us denote by i_{l,x_0} the left involute of ∂K starting at x_0 corresponding to the counterclockwise parameterization of ∂K , by i_{r,x_0} the right involute corresponding to the clockwise parameterization. When one needs to emphasize the dependence on K of involutes, they will be written as i_{l,x_0}^K, i_{r,x_0}^K .

Remark 3.3. Let us notice that if ρ is a plane reflection with respect to a fixed axis then

$$i_{r,x_0}^K = \rho(i_{l,\rho(x_0)}^{\rho(K)}).$$

This relation allows us to prove our results for the left involutes only and to state without proof the analogous results for the right involutes.

Theorem 3.4. Let us fix the initial parameters $x_0, s_0, \theta_0, \vartheta_0$. The left and the right involutes of a plane convex curve starting at $x_0 \in \partial K$, boundary of a C_+^2 plane convex body K with support function h , are parameterized by the value ϑ related to the outer normal \mathbf{n}_ϑ to K , as follows

$$i_{l,x_0}(\vartheta) = h(\vartheta)\mathbf{n}_\vartheta - \left(\int_{\vartheta_0}^{\vartheta} h(\tau)d\tau - \dot{h}(\vartheta_0) \right) \mathbf{t}_\vartheta, \quad \text{for } \vartheta \geq \vartheta_0, \quad (25)$$

$$i_{r,x_0}(\vartheta) = h(\vartheta)\mathbf{n}_\vartheta - \left(\int_{\vartheta_0}^{\vartheta} h(\tau)d\tau - \dot{h}(\vartheta_0) \right) \mathbf{t}_\vartheta, \quad \text{for } \vartheta \leq \vartheta_0. \quad (26)$$

Proof. In the present case there is a 1-1 mapping between ϑ and s ; from (15), it follows

$$s - s_0 = \int_{\vartheta_0}^{\vartheta} h(\tau)d\tau + \dot{h}(\vartheta) - \dot{h}(\vartheta_0);$$

then, changing the variable s with ϑ in (24), with elementary computation, (25) is obtained (since $x'(s) = \mathbf{t}_\vartheta$ and (16) holds). Formula (26) follows from (18) and (21). \square

For an arbitrary convex body K in place of (16), formulas (17),(18) have to be used.

Definition 3.5. Let K be a plane convex body, let

$$x_0 = x(s_0) \in \partial K, \vartheta_0^+ := \vartheta_{l+}(s_0), s_0^+ := s_{l+}(\vartheta_0^+).$$

The left involute of ∂K starting at x_0 will be defined as

$$i_{l,x_0}(\vartheta) = x_l(s_{l+}(\vartheta)) - (s_{l+}(\vartheta) - s_0) \mathbf{t}_\vartheta \quad \text{for } \vartheta \geq \vartheta_0^+; \quad (27)$$

Similarly if $\vartheta_0^- := \vartheta_{r-}(s_0)$, $s_0^- := s_{r-}(\vartheta_0^-)$, the right involute starting at x_0 will be defined as

$$i_{r,x_0}(\vartheta) = x_r(s_{r-}(\vartheta)) + (s_{r-}(\vartheta) - s_0) \mathbf{t}_\vartheta \quad \text{for } \vartheta \leq \vartheta_0^-. \quad (28)$$

From (27), (20) it follows that

$$i_{l,x_0}(\vartheta) = h(\vartheta)\mathbf{n}_\vartheta - \left(\int_{\vartheta_0^+}^{\vartheta} h(\tau)d\tau - \dot{h}_+(\vartheta_0^+) \right) \mathbf{t}_\vartheta - |x_0 - x_l(s_0^+)| \mathbf{t}_\vartheta, \quad \vartheta \geq \vartheta_0^+; \quad (29)$$

Similarly from (28), (21) it follows that

$$i_{r,x_0}(\vartheta) = h(\vartheta)\mathbf{n}_\vartheta - \left(\int_{\vartheta_0^-}^{\vartheta} h(\tau)d\tau - h_-(\vartheta_0^-) \right) \mathbf{t}_\vartheta + |x_0 - x_r(s_0^-)|\mathbf{t}_\vartheta, \quad \vartheta \leq \vartheta_0^-. \quad (30)$$

Let us notice that in (29), (30) the same parameter ϑ is used, but with different range; it turns out that i_l is counterclockwise oriented; instead i_r is clockwise oriented; $x_0 = i_{l,x_0}(\vartheta_0^+) = i_{r,x_0}(\vartheta_0^-)$.

Remark 3.6. *The following facts can be derived from the above equations:*

i) *since h is Lipschitz continuous for every convex body K , then the involute i_{l,x_0} is a rectifiable curve;*

ii) $i_{l,x_0}(\vartheta_0^+) = x_0$ and

$$|i_{l,x_0}(\vartheta) - x_l(s_{l+}(\vartheta))| = s_{l+}(\vartheta) - s_0; \quad (31)$$

iii) *if x is a vertex of ∂K then $i_{l,x_0}(\vartheta)$, for $(\cos \vartheta, \sin \vartheta) \in N_K(x)$, lies on an arc of circle centered at x with radius $s_{l+}(\vartheta) - s_0$;*

iv) *the involute (25) satisfies*

$$i_{l,x_0}(\vartheta + 2\pi) = i_{l,x_0}(\vartheta) - L\mathbf{t}_\vartheta, \quad \forall \vartheta \geq \vartheta_0^+. \quad (32)$$

Lemma 3.7. *The parameterization (29) of the involute i_{l,x_0} is 1-1 in the interval $[\vartheta_0^+, \vartheta_0^+ + 2\pi)$; moreover, except for at most a finite or countable set \mathfrak{F} of values ϑ_i , $i = 1, 2, \dots$ (corresponding to the 1-faces F_{θ_i} of ∂K), i_{l,x_0} is differentiable and:*

$$\frac{d}{d\vartheta} i_{l,x_0}(\vartheta) = (s_{l+}(\vartheta) - s_0)\mathbf{n}_\vartheta \quad \text{for } \vartheta > \vartheta_0^+, \vartheta \notin \mathfrak{F}; \quad (33)$$

furthermore i_{l,x_0} has left and right derivative with common direction \mathbf{n}_ϑ at $\vartheta = \vartheta_i \in \mathfrak{F}$.

Proof. By differentiating (29) and using (20), the equality (33) is proved. Similar argument, at $\vartheta = \vartheta_i \in \mathfrak{F}$, proves that \mathbf{n}_ϑ is the common direction of the left and right derivatives. \square

Remark 3.8. *Let $\vartheta \rightarrow i_{l,x_1}(\vartheta)$, $\vartheta \rightarrow i_{l,x_2}(\vartheta)$, $x_i = x(s_i)$, $i = 1, 2$ be left involutes of K . Since*

$$i_{l,x_2}(\vartheta) - i_{l,x_1}(\vartheta) = (s_2 - s_1)\mathbf{t}_\vartheta, \quad \text{for } \vartheta > \max\{\vartheta_l^+(s_2), \vartheta_l^+(s_1)\},$$

then they will be called parallel curves. Moreover, by (32), $i_{l,x_0}(\vartheta)$ and $i_{l,x_0}(\vartheta + 2\pi)$ will also be called parallel.

Theorem 3.9. *If $d\sigma$ is the arc element of the involute i_{l,x_0} then $\vartheta \rightarrow \sigma(\vartheta)$ is continuous and invertible in $\vartheta \geq \vartheta_0^+$ with continuous inverse $[0, +\infty) \ni \sigma \rightarrow \vartheta(\sigma)$. Moreover*

$$d\sigma = (s_{l+}(\vartheta) - s_0) d\vartheta \quad \text{for } \vartheta \geq \vartheta_0^+, \vartheta \notin \mathfrak{F}; \quad (34)$$

the involute is a convex curve with positive curvature a.e.

$$\frac{d\vartheta}{d\sigma} = \frac{1}{(s_{l+}(\vartheta) - s_0)} \quad \text{for } \vartheta > \vartheta_0^+, \vartheta \notin \mathfrak{F}, \quad (35)$$

$\sigma \rightarrow i_{l,x_0}(\vartheta(\sigma))$ is C^1 everywhere and

$$\frac{d}{d\sigma} i_{l,x_0} = \mathbf{n}_{\vartheta(\sigma)}. \quad (36)$$

Moreover the following properties hold.

i) *For every $\sigma > 0$ the right derivative*

$$\left(\frac{d\vartheta}{d\sigma}\right)^+ = \frac{1}{s_{l+}(\vartheta(\sigma)) - s_0}$$

exists everywhere and it is a decreasing cadlag function;

ii) $\frac{d}{d\sigma} i_{l,x_0}$ *has everywhere right derivative given by*

$$\left(\frac{d^2}{d\sigma^2} i_{l,x_0}\right)^+ = -\frac{1}{s_{l+}(\vartheta(\sigma)) - s_0} \mathbf{t}_{\vartheta(\sigma)}.$$

Theorem 3.10. *Let $K^{(n)}$ be a sequence of plane convex bodies which converges uniformly to K , $x^{(n)} \in \partial K^{(n)}$, $x^{(n)} \rightarrow x_0$; then the corresponding sequences of left involutes $i_{l,x^{(n)}}^{K^{(n)}}$ converge uniformly to i_{l,x_0} in compact subsets of $[\vartheta_0^+, +\infty)$; moreover the corresponding sequence of their derivatives (with respect to the arc length) converges uniformly to $\frac{d}{d\sigma} i_{l,x_0}$.*

Proof. By Remark 2.8 the sequence of functions s_{l+}^n converge to s_{l+} . From (34) the arclengths of the left involutes $i_{l,x^{(n)}}^{K^{(n)}}$

$$\sigma^{(n)}(\vartheta) = \int_{\vartheta_0}^{\vartheta} \left(s_{l+}^{(n)}(\vartheta) - s_0^{(n)} \right) d\vartheta$$

converges uniformly in compact subsets of $[\vartheta_0^+, +\infty)$ to the arc length $\sigma(\vartheta)$ of i_{l,x_0} ; from (36) the same fact holds for their derivatives. \square

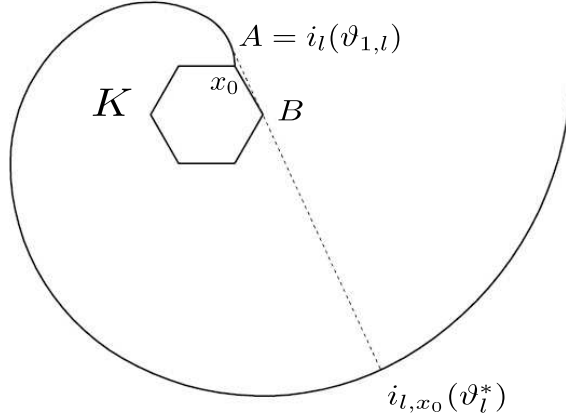


Figure 2: Left involute of an exagon

Let us consider the arc of the involute:

$$\eta := \{i_{l,x_0}(\vartheta) : \vartheta_0^+ \leq \vartheta \leq \vartheta_0^+ + 3\pi/2\}$$

and the set valued map F (Definition 2.7). Let

$$Q = \bigcup_{\vartheta_0^+ \leq \vartheta \leq \vartheta_0^+ + 3\pi/2} \{\lambda F(\theta) + (1 - \lambda)i_{l,x_0}(\vartheta), \quad 0 \leq \lambda \leq 1\}, \quad \theta = (\cos \vartheta, \sin \vartheta),$$

the union of segments joining the points of η with the corresponding points on ∂K .

Definition 3.11. *If the tangent sector $T(x_0)$ to K has an opening less or equal than $\pi/2$ as in Fig.1, then $Q \cup K$ is convex; let us define*

$$\vartheta_l^* = \vartheta_0^+ + 3\pi/2.$$

If $Q \cup K$ is not convex then let us consider $co(Q \cup K)$. Let us notice that $\partial co(Q \cup K) \setminus \partial(Q \cup K)$ is an open segment with end points A, B , with $A \in \eta$, $B \in \partial K$. Let us define ϑ_l^ , with $\vartheta_0^+ + 3\pi/2 \leq \vartheta_l^* < \vartheta_0^+ + 2\pi$ such that (see Fig.2) $\theta_l^* = (\cos \vartheta_l^*, \sin \vartheta_l^*)$ is orthogonal to AB , $B \in F(\theta_l^*)$. Let $\vartheta_{1,l}$ be the smallest $\theta > \theta_0^+$ satisfying $A = i_{l,x_0}(\vartheta_{1,l})$. Clearly $\vartheta_l^* = \vartheta_{1,l} + \frac{3}{2}\pi$.*

For the right involutes a value ϑ_r^ is defined similarly, with $\vartheta_0^- - 2\pi < \vartheta_r^* \leq \vartheta_0^- - 3\pi/2$, such that the line orthogonal to θ_r^* supporting K at $F(\theta_r^*)$ is tangent to the right involute at $i_{r,x_0}(\vartheta_{1,r})$, see Fig.3 where $F(\theta_r^*)$ is the point $x(s_{r-}(\vartheta_r^*))$, written as $x(\vartheta_r^*)$ for short.*

Theorem 3.12. *Let $i_l := i_{l,x_0}$ be the left involute starting at x_0 on the boundary of a plane convex body K , then:*

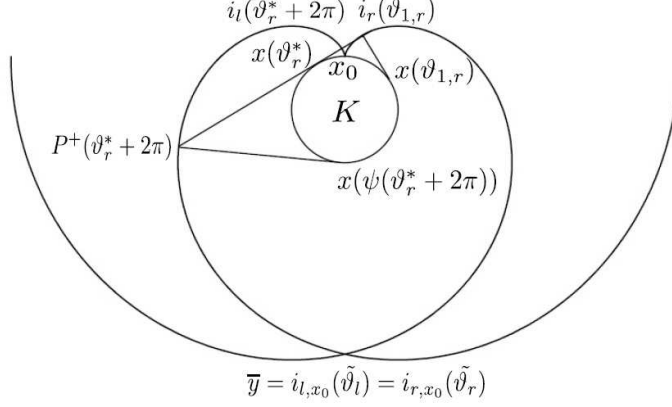


Figure 3: Involute of a circumference

- i) the left involute $\vartheta \rightarrow i_l(\vartheta)$ has the distancing from K property for $\vartheta \geq \vartheta_0^+$, but is not a SDC for $\vartheta \geq \vartheta_l^*$;
- ii) the curve $\vartheta \in [\vartheta_0^+, \vartheta_l^*] \rightarrow i(\vartheta)$ is a SDC;
- iii) for $y \in \text{Int}(K)$ the distance function $J_y(\vartheta) = |i_l(\vartheta) - y|$ is strictly increasing for $\vartheta \geq \vartheta_0^+$;
- iv) if $y \in \partial K$, then $J_y(\vartheta)$ is not decreasing for $\vartheta \geq \vartheta_0^+$ and $\frac{d}{d\vartheta}J > 0$ for $(\cos \vartheta, \sin \vartheta) \notin N_K(y)$.

Proof. As i_l is rectifiable, then the function $J_y^2(\vartheta) = |i_l(\vartheta) - y|^2$ is an absolutely continuous function for $\vartheta \geq \vartheta_0^+$, and from (33) for $\vartheta \notin \mathfrak{F}$

$$\begin{aligned} \frac{1}{2} \frac{d}{d\vartheta} J_y^2 &= \left\langle \frac{d}{d\vartheta} i_l, i_l(\vartheta) - y \right\rangle = \langle (s_l(\vartheta) - s_0) \mathbf{n}_\vartheta, x_l(s_{l+}(\vartheta)) + (s_{l+}(\vartheta) - s_0) \mathbf{t}_\vartheta - y \rangle = \\ &= (s_{l+}(\vartheta) - s_0) \langle \mathbf{n}_\vartheta, x_l(s_{l+}(\vartheta)) - y \rangle \geq 0; \end{aligned}$$

the last inequality holds since \mathbf{n}_ϑ is the outer normal to ∂K at $x_l(s_{l+}(\vartheta))$. Moreover the previous inequality is strict for all ϑ if $y \in \text{Int}(K)$, it is also a strict inequality for $y \in \partial K$ and $y \notin F(\theta)$. This proves iii) and iv). Then i) follows from iii) and Definition 1.1 of distancing from K property for a curve. To prove ii) let us recall that a SDC satisfies (1); then one has to prove that the angle at $i_l(\vartheta)$ between the vector $i_l(\vartheta) - i_l(\tau)$, $\vartheta_0^+ < \tau < \vartheta \leq \vartheta_l^*$, and \mathbf{n}_ϑ , the tangent vector at $i_l(\vartheta)$, is greater or equal than $\pi/2$; this is equivalent to show that the half line r_ϑ through $i_l(\vartheta)$ and $x(s_{l+}(\vartheta))$ orthogonal to \mathbf{n}_ϑ supports at $i_l(\vartheta)$ the arc of i_l from x_0 to $i_l(\vartheta)$. By Definition 3.11 this is the case for all ϑ between ϑ_0^+ and ϑ_l^* . \square

Corollary 3.13. *The left involute $\vartheta \rightarrow i_{l,x_0}(\vartheta)$ of the boundary of a plane convex body K is a self-distancing curve from K for $\vartheta \in [\vartheta_0^+, \vartheta_l^*]$; similarly the right involute (30) is a self-contracting curve from K for $\vartheta \in [\vartheta_r^*, \vartheta_0^-]$.*

Proof. From i) of Theorem 3.12 the left involute is a curve such that the distance of its points from all $y \in K$ is not decreasing; ii) of the same theorem proves that it is a SDC. Let us recall that a self-contracting curve is a self-distancing curve with opposite orientation. \square

Theorem 3.14. *Let K be a plane convex body not reduced to a single point and let $x_0, s_0, \theta_0, \vartheta_0$ be the initial parameters. Let $[\vartheta_0^+, \vartheta_0^+ + 2\pi] \ni \vartheta \rightarrow i_l(\vartheta)$ be an arc of the left involute starting at x_0 , $[\vartheta_0^- - 2\pi, \vartheta_0^-] \ni \vartheta \rightarrow i_r(\vartheta)$ be an arc of the right involute ending at x_0 ; then there exists only one point $\bar{y} \neq x_0$ which belongs to both arcs and*

$$\bar{y} = i_l(\tilde{\vartheta}_l) = i_r(\tilde{\vartheta}_r), \quad (37)$$

with

$$\begin{aligned} \vartheta_0^- &\leq \vartheta_r^* + 2\pi < \tilde{\vartheta}_l < \vartheta_0^+ + 3\pi/2 \leq \vartheta_l^*, \\ \vartheta_r^* &\leq \vartheta_0^- - 3\pi/2 < \tilde{\vartheta}_r < \vartheta_l^* - 2\pi \leq \vartheta_0^+. \end{aligned}$$

Proof. For simplicity, first let us prove the existence of \bar{y} assuming that $K \in C_+^2$. With the assumed conditions, $\mathbb{R} \ni \vartheta \rightarrow x(\vartheta) := x(\theta)$ defined by (16) is a parameterization of ∂K .

Let $\vartheta \in [\vartheta_0, \vartheta_0 + 2\pi]$ and let $P^+(\vartheta)$ be the first common point of the half line $\{x(\vartheta) + \lambda \mathbf{t}_\vartheta, \lambda > 0\}$ and of i_l . Moreover, let $[\vartheta_0, \vartheta_l^*] \ni \vartheta \rightarrow \psi(\vartheta)$ be the function satisfying

$$P^+(\vartheta) = i_l(\psi(\vartheta)). \quad (38)$$

Let

$$\phi(\vartheta) := |P^+(\vartheta) - i_l(\vartheta)|.$$

First the following sentence will be proved:

Claim 1. $P^+(\bar{\vartheta}_l)$ belongs to i_r iff the equality

$$\phi(\bar{\vartheta}_l) = L \quad (39)$$

holds for some $\bar{\vartheta}_l \in [\vartheta_0, \vartheta_0 + 2\pi]$, $L = |\partial K|$.

Proof of Claim 1.

If (39) holds, then

$$\begin{aligned} |P^+(\bar{\vartheta}_l) - x(\bar{\vartheta}_l)| &= |P^+(\bar{\vartheta}_l) - i_l(\bar{\vartheta}_l)| - |i_l(\bar{\vartheta}_l) - x(\bar{\vartheta}_l)| = \\ &= L - |\text{arc}^+(x_0, x(\bar{\vartheta}_l))| = |\text{arc}^+(x(\bar{\vartheta}_l), x_0)| = |\text{arc}^-(x_0, x(\bar{\vartheta}_l - 2\pi))|. \end{aligned}$$

Thus

$$P^+(\bar{\vartheta}_l) = x(\bar{\vartheta}_l) + |\text{arc}^+(x(\bar{\vartheta}_l), x_0)| \mathbf{t}_{\bar{\vartheta}_l} = x(\bar{\vartheta}_l - 2\pi) + |\text{arc}^-(x_0, x(\bar{\vartheta}_l - 2\pi))| \mathbf{t}_{\bar{\vartheta}_l - 2\pi} = i_r(\bar{\vartheta}_l - 2\pi).$$

Thus $P^+(\overline{\vartheta_l})$ is on both arcs of involutes and the other way around.

Our aim is to prove that there exists $\overline{\vartheta_l} \in [\vartheta_0, \vartheta_0 + 3\pi/2]$ such that (39) holds. For this goal we prove next Claim 2 and Claim 3.

Claim 2: The following facts hold in $[\vartheta_0, \vartheta_l^*]$:

- i) ψ is continuously differentiable and $\psi' > 0$,
- ii) $\phi' > 0$.

Proof of Claim 2.

Let us prove that \mathbf{n}_ϑ and $\mathbf{n}_{\psi(\vartheta)}$ satisfy

$$\langle \mathbf{n}_\vartheta, \mathbf{n}_{\psi(\vartheta)} \rangle < 0. \quad (40)$$

Let us consider the triangle with vertices $x(\vartheta), i_l(\psi(\vartheta)), x(\psi(\vartheta))$. As

$$|i_l(\psi(\vartheta)) - x(\psi(\vartheta))| = |\text{arc}^+(x_0, x(\psi(\vartheta)))| \geq |\text{arc}^+(x(\vartheta), x(\psi(\vartheta)))| \geq |x(\psi(\vartheta)) - x(\vartheta)|,$$

the angle between $x(\psi(\vartheta)) - P^+(\vartheta)$ and $x(\vartheta) - P^+(\vartheta)$ is acute and the angle between \mathbf{n}_ϑ and $\mathbf{n}_{\psi(\vartheta)}$ is obtuse. Thus (40) follows. By definition, $\psi(\vartheta)$ solves (38), thus $\psi(\vartheta)$ is the implicit solution to

$$\langle i_l(\psi(\vartheta)) - x(\vartheta), \mathbf{n}_\vartheta \rangle = 0. \quad (41)$$

As

$$\left\langle \frac{d}{d\psi} i_l(\psi), \mathbf{n}_\vartheta \right\rangle = (s(\psi) - s_0) \langle \mathbf{n}_\psi, \mathbf{n}_\vartheta \rangle$$

is negative by (40), then by Dini's Theorem equation (41) has a solution $\psi(\theta)$ satisfying

$$(s(\psi) - s_0) \langle \mathbf{n}_\psi, \mathbf{n}_\vartheta \rangle \psi'(\vartheta) + \langle i_l(\psi(\vartheta)) - x(\vartheta), \mathbf{t}_\vartheta \rangle = 0.$$

As $i_l(\psi(\vartheta)) - x(\vartheta) = \lambda \mathbf{t}_\vartheta$ ($\lambda > 0$) and (40) holds, then $\psi' > 0$, ψ is strictly increasing and continuously differentiable.

Let us prove (ii). The formula

$$\frac{d}{d\vartheta} |i_l(\vartheta) - i_l(\psi(\vartheta))|^2 = 2 \langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\vartheta) - \frac{d}{d\vartheta} i_l(\psi(\vartheta)) \rangle \quad (42)$$

holds. Let us notice that $i_l(\vartheta) - i_l(\psi(\vartheta))$ is parallel to \mathbf{t}_ϑ ; thus by (33)

$$\langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\vartheta) \rangle = 0.$$

On the other hand

$$\begin{aligned} -\langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\psi(\vartheta)) \rangle &= -\langle -s(\vartheta) \mathbf{t}_\vartheta - \lambda \mathbf{t}_\vartheta, (s(\psi(\vartheta)) - s_0) \mathbf{n}_{\psi(\vartheta)} \rangle \psi' \\ &= (s(\vartheta) + \lambda)(s(\psi(\vartheta)) - s_0) \langle \mathbf{t}_\vartheta, \mathbf{n}_{\psi(\vartheta)} \rangle \psi'. \end{aligned}$$

As the angle between \mathbf{t}_ϑ and $\mathbf{n}_{\psi(\vartheta)}$ is acute, then last term in the above equalities is positive; thus the derivative in the left hand side of (42) is positive and (ii) of Claim 2 follows.

Claim 3: In the interval $[\vartheta_0, \vartheta_l^*]$ the function ϕ has values smaller than L and greater than L .

Proof of Claim 3.

The angles ϑ_r^* , and $\vartheta_{1,r}$ has been introduced in Definition 3.11. For simplicity $x(s_{r-}(\vartheta_{1,r}))$ will be denoted with $x(\vartheta_{1,r})$. Let us consider the convex set bounded by $\text{arc}^+(x(\psi(\vartheta_r^* + 2\pi)), x(\vartheta_{1,r}))$ and by the polygonal line with vertices $x(\vartheta_{1,r}), i_r(\vartheta_{1,r}), P^+(\vartheta_r^* + 2\pi), x(\psi(\vartheta_r^* + 2\pi))$, see Fig.3.

Clearly the inequalities

$$\begin{aligned} |i_r(\vartheta_{1,r}) - P^+(\vartheta_r^* + 2\pi)| &< |i_r(\vartheta_{1,r}) - x(\vartheta_{1,r})| + |\text{arc}^-(x(\vartheta_{1,r}), x(\psi(\vartheta_r^* + 2\pi)))| + |x(\psi(\vartheta_r^* + 2\pi)) - P^+(\vartheta_r^* + 2\pi)| \\ &= |\text{arc}^-(x_0, x(\vartheta_{1,r}))| + |\text{arc}^-(x(\vartheta_{1,r}), x(\psi(\vartheta_r^* + 2\pi)))| + |\text{arc}^-(x(\psi(\vartheta_r^* + 2\pi)), x_0)| = L \end{aligned}$$

hold. As

$$\phi(\vartheta_r^* + 2\pi) = |i_l(\vartheta_r^* + 2\pi) - P^+(\vartheta_r^* + 2\pi)| < |i_r(\vartheta_{1,r}) - P^+(\vartheta_r^* + 2\pi)|,$$

using the previous inequalities, one obtains

$$\phi(\vartheta_r^* + 2\pi) < L.$$

Let us show now that

$$\phi(\vartheta_0 + 3\pi/2) > L \tag{43}$$

holds.

Let ρ be the half line with origin x_0 and direction $-\mathbf{t}_{\vartheta_0}$; $\rho - \{x_0\}$ crosses the arc i_r in a first point $y_1 = i_r(\alpha_1)$, with $\alpha_1 < \vartheta_0 - \pi/2$. Then

$$r := |x_0 - y_1| < |y_1 - x(\alpha_1)| + |\text{arc}^-(x(\alpha_1), x_0)| = L.$$

The half line ρ meets the arc i_l in a point y_2 and $|y_2 - x_0| = L$.

Property (iii) of Theorem 3.12 implies that the arc D of the left involute after y_2 lies outside of the circle centered in x_0 and with radius L . The similar property for the right involute implies that the arc C of the right involute joining x_0 to y_1 lies in the circle with center x_0 and radius r ; thus the straight line tangent to K at $x(\vartheta_0 + 3\pi/2)$ meets the arc C in $i_r(\vartheta_0 - \pi/2)$ and D in $P^+(\vartheta_0 + 3\pi/2)$. Therefore

$$\begin{aligned} \phi(\vartheta_0 + 3\pi/2) &= |i_l(\vartheta_0 + 3\pi/2) - P^+(\vartheta_0 + 3\pi/2)| = \\ &= |i_l(\vartheta_0 + 3\pi/2) - x(\vartheta_0 + 3\pi/2)| + |x(\vartheta_0 + 3\pi/2) - P^+(\vartheta_0 + 3\pi/2)| > \\ &> |i_l(\vartheta_0 + 3\pi/2) - x(\vartheta_0 + 3\pi/2)| + |x(\vartheta_0 + 3\pi/2) - i_r(\vartheta_0 - \pi/2)| = \\ &= |\text{arc}^+(x_0, x(\vartheta_0 + 3\pi/2))| + |\text{arc}^-(x_0, x(\vartheta_0 + 3\pi/2))| = L. \end{aligned}$$

(43) is proved.

The intermediate values theorem implies that there exists $\overline{\vartheta}_l \in [2\pi + \vartheta_r^*, \vartheta_0 + 3\pi/2]$ such that (39) holds. Claim 1 implies that

$$P^+(\overline{\vartheta}_l) = i_l(\psi(\overline{\vartheta}_l)) = i_r(\overline{\vartheta}_l - 2\pi),$$

so the right involute and the left involute meet each other in one point and (37) is proved with $\tilde{\vartheta}_l = \psi(\overline{\vartheta}_l)$, $\tilde{\vartheta}_r = \overline{\vartheta}_l - 2\pi$.

By approximation argument the same result holds for an arbitray convex body K .

Let us prove now that the point \overline{y} is unique. Let us argue by contradiction. Let P, Q be two distinct points on $i_l \cap i_r$, with $P \prec Q$ on i_l and i_r ; then since i_l is a distancing curve from x_0 :

$$|P - x_0| \leq |Q - x_0|,$$

and since i_r is a contracting curve to x_0 :

$$|P - x_0| \geq |Q - x_0|.$$

Therefore all the points on the arc of i_l and of i_r between P and Q have the same distance from x_0 ; thus, between P and Q , i_l and i_r (arc of involutes of a same convex body K) coincide with the same arc of circle centered at x_0 , this implies that K reduce to the point x_0 , which is not possible for the assumption. \square

Definition 3.15. Let $z \notin K$. Let $z_l(z_r) \in \partial K$ on the contact set on the “left” (right) support line to K through z . If the contact set is a 1-face on these support lines, then z_l and z_r are identified as the closest ones to z . The triangle zz_lz_r is counterclockwise oriented.

Theorem 3.16. For every $\xi \in \partial K$ let us consider the left involutes $i_{l,\xi}$ and the right involutes $i_{r,\xi}$ parameterized by their arc length σ . The maps

$$\partial K \times (0, +\infty) \ni (\xi, \sigma) \rightarrow i_{l,\xi}(\theta(\sigma)) \in \mathbb{R}^2 \setminus K,$$

$$\partial K \times (0, +\infty) \ni (\xi, \sigma) \rightarrow i_{r,\xi}(\theta(\sigma)) \in \mathbb{R}^2 \setminus K$$

are 1-1 maps.

Proof. Assume, in the proof, that $x_0 \in \partial K, \theta_0 \in G(x_0), \vartheta_0, s_0$ are fixed. Let $z \notin K$. The tangent sector to the cap body K^z with vertex z has two maximal segments zz_l, zz_r on the sides that do not meet K (except at the end points z_l, z_r). Let ϑ_l such that $z_l = x_l(s_{l+}(\vartheta_l))$, and let \overline{s} such that

$$|z - z_l| = s_{l+}(\vartheta_l) - \overline{s}.$$

Let $\xi_l = x_l(\bar{s})$, let $\bar{\vartheta} = \vartheta_l^+(\bar{s})$. From (31) and from the definition of left involute (27) (with ξ_l in place of x_0 , $\bar{\vartheta}$ in place of ϑ_0^+ , \bar{s} in place s_0)

$$z = i_{l,\xi_l}(\vartheta_l)$$

holds; thus the map $(\xi, \sigma) \rightarrow i_{l,\xi}(\sigma)$ is surjective. Moreover the map it is also injective, since the left involutes don't cross each other since they are parallel (see Remark 3.8). Similar proof holds for the right involutes. \square

Let $\xi_l = x_l(\bar{s})$ be the starting point of the left involute i_{l,ξ_l} through z , defined in the previous theorem; similarly let ξ_r be the starting point of the right involute i_{r,ξ_r} through z . Let us notice that i_{l,ξ_l} and i_{r,ξ_r} meet each other in a countable ordered set of points.

3.1 \mathfrak{J} -fence and \mathfrak{G} -fence

Definition 3.17. Let K be a convex body in \mathbb{R}^2 , $|\partial K| > 0$, $x_0 \in \partial K$, $\theta_0 \in G(x_0)$, $\theta_0 = (\cos \vartheta_0, \sin \vartheta_0)$, $s_0 \in \mathbb{R}$. Let $i_l := i_{l,x_0}$, $i_r := i_{r,x_0}$. Let

$$\bar{y} = i_l(\tilde{\vartheta}_l) = i_r(\tilde{\vartheta}_r) \in \mathbb{R}^2 \setminus K,$$

be the first point where the two involutes cross each other (see Theorem 3.14). Let us define

$$\mathfrak{J}_l(K, x_0) := \{y \in \mathbb{R}^2 : y = tx_0 + (1-t)i_l(\vartheta), \quad 0 \leq t \leq 1, \vartheta_0^+ \leq \vartheta \leq \tilde{\vartheta}_l\},$$

$$\mathfrak{J}_r(K, x_0) := \{y \in \mathbb{R}^2 : y = tx_0 + (1-t)i_r(\vartheta), \quad 0 \leq t \leq 1, \tilde{\vartheta}_r \leq \vartheta \leq \vartheta_0^-\},$$

$$\mathfrak{J}(K, x_0) := (\mathfrak{J}_l(K, x_0) \cup \mathfrak{J}_r(K, x_0)) \setminus \text{Int}(K).$$

$\mathfrak{J}(K, x_0)$ will be called the \mathfrak{J} -fence of K at x_0 .

Let us notice that $\mathfrak{J}_l(K, x_0)$ and $\mathfrak{J}_r(K, x_0)$ are two convex bodies with in common the segment $x_0\bar{y}$ only.

From Theorem 3.16 the starting point ξ_l (ξ_r) of a left(right) involute is uniquely determined from any point $z \notin K$ of the involute. The arc of the points on the left (right) involute between the starting point and z will be denoted by i_{l,ξ_l}^z (i_{r,ξ_r}^z), or i_l^z (i_r^z) for short. For $y \preceq w$ let us denote with $i_l^{y,w}$ ($i_r^{y,w}$) the oriented arc of the left (right) involute between y and w .

Let us introduce now other regions which are bounded by left and right involutes.

Let us fix the initial parameters $x_0, s_0, \theta_0, \vartheta_0$.

Definition 3.18. Given $z \in \mathbb{R}^2 \setminus K$, let $i_l = i_{l,\xi_l}$ ($i_r = i_{r,\xi_r}$) be the left (right) involute through z with starting point ξ_l (ξ_r) and let $z_l(z_r) \in \partial K$ be as in Definition 3.15. Let $\vartheta_{\xi_l}^+$ satisfying $x_l(s_{l+}(\vartheta_{\xi_l}^+)) = \xi_l$. Let $\vartheta_l >$

$\vartheta_{\xi_l}^+$ be the smallest angle for which $x_l(s_{l+}(\vartheta_l)) = z_l$. Let us consider the parameterization (27); let us define

$$\mathfrak{G}_l(K, z) := \{tx_l(s_{l+}(\vartheta)) + (1-t)i_l(\vartheta), \quad 0 < t < 1, \quad \vartheta_{\xi_l}^+ < \vartheta < \vartheta_l\}. \quad (44)$$

If i_l^z does not cross the open segment zz_l , the region $\mathfrak{G}_l(K, z)$ is an open set bounded by the convex arc of left involute i_l^z , the segment zz_l and the convex arc of ∂K : $\text{arc}^+(\xi_l, z_l)$; otherwise let w be the nearest point to z where i_l^z crosses the open segment zz_l ; the region $\mathfrak{G}_l(K, z)$ is an open set bounded by the arc $i_l^{w,z}$, the segment wz and ∂K . Similarly let us define $\mathfrak{G}_r(K, z)$.

$\mathfrak{G}_l(K, z), \mathfrak{G}_r(K, z)$ are open and bounded sets. Let us define:

$$\mathfrak{G}(K, z) := \text{Int}(\text{cl}(\mathfrak{G}_l(K, z) \cup \mathfrak{G}_r(K, z))). \quad (45)$$

$\mathfrak{G}(K, z)$ is an open, bounded, connected set. $\mathfrak{G}(K, z)$ will be called the \mathfrak{G} -fence of K at z .

Remark 3.19. If z is the first crossing point of i_l and i_r and $\xi_l = \xi_r$, then $\mathfrak{G}(K, z) = \text{Int}(\mathfrak{J}(K, \xi_l))$.

Let us conclude this section with the following result, which follows from Theorem 3.10.

Theorem 3.20. Let K be limit of a sequence of convex bodies $K^{(n)}$, $x_0 = \lim x_0^{(n)}$, $x_0^{(n)} \in \partial K^{(n)}$. Then

$$\mathfrak{J}(K^{(n)}, x_0^{(n)}) \rightarrow \mathfrak{J}(K, x_0).$$

Moreover if $z \notin K$, $z = \lim z^{(n)}$, $z^{(n)} \notin K^{(n)}$, then

$$\text{cl}(\mathfrak{G}(K^{(n)}, z^{(n)})) \rightarrow \text{cl}(\mathfrak{G}(K, z)).$$

4 Bounding regions for SDC in the plane

Let us assume that x_0 is the end point of one of the following sets

- a) a steepest descent curve γ , satisfying (1) and (3);
- b) γ^K : a self-distancing curve from a convex body K , see Definition 2.2.

The following questions arise: can one extend γ , γ^K beyond x_0 ? Which regions delimit that extension? Which regions are allowed and which are forbidden?

Lemma 4.1. Let $z \in \mathbb{R}^2 \setminus K$. If $u \in \mathfrak{G}_l(K, z)$ then the arc i_l^u of the left involute to K ending at u , is contained in $\mathfrak{G}_l(K, z)$. Similarly if $u \in \mathfrak{G}_r(K, z)$, then $i_r^u \subset \mathfrak{G}_r(K, z)$.

Proof. Since $u \in \mathfrak{G}_l(K, z)$, by (44) there exist $\overline{\vartheta}_l \in (\vartheta_{\xi_l}^+, \vartheta_l)$, $\tau \in (0, 1)$ such that

$$u = \tau x_l(s_{l+}(\overline{\vartheta}_l)) + (1 - \tau) i_l(\overline{\vartheta}_l).$$

Then the arc i_l^u is parallel to an arc of the left involute i_l (through z) for $\vartheta \in (\vartheta_{\xi_l}^+, \overline{\vartheta}_l)$. Then any left tangent segment to K from a point of i_l^u is contained in the left tangent segment from the corresponding point of i_l^z . \square

Lemma 4.2. *Let $z \in \mathbb{R}^2 \setminus K$ and let $u \in \mathfrak{G}_l(K, z)$. There are two possible cases:*

- i) *if the right involute ending at u does not cross the tangent segment $z_l z$ or it crosses $z_l z$ at a point $q \in \mathfrak{G}_l(K, z)$, then in both cases $i_r^u \subset \mathfrak{G}_l(K, z)$;*
- ii) *if the right involute ending at u crosses the tangent segment $z_l z$ at a point $q \in z_l z \cap \partial \mathfrak{G}_l(K, z)$, then $i_r^{q,u} \setminus \{q\} \subset \mathfrak{G}_l(K, z)$.*

Proof. Since the starting point $\xi_r(u)$ of the right involute ending at u is on ∂K , the distance from $\xi_r(u)$ to a point of the left involute i_l^z is not decreasing, see iv) of Theorem 3.12; similarly the distance from $\xi_r(u)$ to a point of i_r^u is not decreasing. In the case i) the arc i_r^u has its end points in $\mathfrak{G}_l(K, z)$ and by the above distance property it can not cross two times the left involute, then it can not cross the boundary of $\mathfrak{G}_l(K, z)$, therefore $i_r^u \subset \mathfrak{G}_l(K, z)$; similarly in the case ii) the arc $i_r^{q,u}$ can not cross the boundary of $\mathfrak{G}_l(K, z)$ at most than in q ; therefore all the points of this arc, except than q , belong to $\mathfrak{G}_l(K, z)$. \square

From the previous lemma it follows that

Theorem 4.3. *Let $z \notin K$. The following inclusions hold:*

- a) *if $u \in \mathfrak{G}_l(K, z)$, then*

$$cl(\mathfrak{G}_l(K, u)) \setminus \partial K \subset \mathfrak{G}_l(K, z); \quad (46)$$

- b) *if $u \in \mathfrak{G}_r(K, z)$, then*

$$cl(\mathfrak{G}_r(K, u)) \setminus \partial K \subset \mathfrak{G}_r(K, z); \quad (47)$$

- c) *if $u \in \mathfrak{G}(K, z)$, then*

$$cl(\mathfrak{G}(K, u)) \setminus \partial K \subset \mathfrak{G}(K, z). \quad (48)$$

Proof. By Lemma 4.1 the left involute that bounds $\mathfrak{G}_l(K, u)$ is inside $\mathfrak{G}_l(K, z)$, then (46) is proved. Inclusion (47) is proved similarly. Let $u \in \mathfrak{G}(K, z) = \text{Int}(cl(\mathfrak{G}_l(K, z) \cup \mathfrak{G}_r(K, z)))$ and let us consider $u \in \mathfrak{G}_l(K, z)$, then in

case i) of Lemma 4.2 also the open arc of the right involute i_r^u is inside $\mathfrak{G}_l(K, z) \subset \mathfrak{G}(K, z)$. Besides $i_l^u \subset \partial \mathfrak{G}_l(K, u)$, then (48) is trivial. In case ii) of Lemma 4.2 the open arc $i_r^{q,u}$ is inside $\mathfrak{G}_l(K, z)$. On the other hand q is inside $\mathfrak{G}_r(K, z)$ and by (47) the arc $i_r^q \subset i_r^u$ is in $\mathfrak{G}_r(K, z) \subset \mathfrak{G}(K, z)$. Similar arguments hold if $u \in \mathfrak{G}_r(K, z)$. Then (48) holds in this case too. \square

Lemma 4.4. *Let $w \notin K$. Let η be polygonal deleted SDC_K with end point $y \in \mathfrak{G}(K, w)$. Then*

$$\eta \subset \mathfrak{G}(K, w), \quad (49)$$

and

$$\eta \subset cl(\mathfrak{G}(K, y)). \quad (50)$$

Proof. To prove (49), let us assume, by contradiction, that η has a point $z \notin \mathfrak{G}(K, w)$. With no loss of generality it can be assumed that $z \in \partial \mathfrak{G}(K, w)$ and

$$\eta \setminus \eta_z \subset \mathfrak{G}(K, w).$$

Then, z is the end point of a segment zw_i , where $w_i \in \mathfrak{G}(K, w) \cap \eta$ and $z \prec w_i$ on η . As $z \in \partial \mathfrak{G}(K, w)$, then there exists an involute through z which is a piece of the boundary of $\mathfrak{G}(K, w)$ (to fix the ideas it is assumed that it is the left involute i_l). Let us consider $z_l \in \partial K$ so that the tangent vector \mathbf{t}_z to i_l at z satisfies

$$\langle \mathbf{t}_z, z - z_l \rangle = 0.$$

As w_i is inside the orthogonal angle centered in z with sides \mathbf{t}_z and $z_l - z$, then

$$\langle w_i - z, z - z_l \rangle < 0.$$

Then as for $\varepsilon > 0$ sufficiently small, $z_\varepsilon := z + \varepsilon(w_i - z) \in \eta_{w_i}$ and at z_ε the curve η has tangent vector $w_i - z$ that satisfies

$$\langle w_i - z, z_\varepsilon - z_l \rangle < 0,$$

contradicting the fact that η_w has the distancing from K property (4). This proves (49).

If $w_n \rightarrow y$, with $y \in \mathfrak{G}(K, w_n)$, also the inclusions

$$\eta \subset cl(\mathfrak{G}(K, w_n))$$

hold. Then (50) is obtained by the approximation Theorem 3.20. \square

Theorem 4.5. *Let K be a convex body and let γ^K be SDC_K , $w \in \gamma$, $w \notin K$. Then*

$$\gamma_w^K \subset cl(\mathfrak{G}(K, w)). \quad (51)$$

Proof. Let us choose a sequence $\{w_n\}, w_n \in \gamma^K, w_n \preceq w, w_n \rightarrow w$. Let us fix the arc $\gamma_{w_n}^K$. By [9, Theorem 6.16], $\gamma_{w_n}^K$ is limit of SDC_K polygonals with end point w_n . From Lemma 4.4, these polygonals are enclosed in $cl(\mathfrak{G}(K, w_n))$; then

$$\gamma_{w_n}^K \subset cl(\mathfrak{G}(K, w_n))$$

holds too. The inclusion (51) is now obtained from the limit of the previous inclusions and by the approximation Theorem 3.20. \square

Theorem 4.6. *Let K be a convex body not reduced to a point. If γ^K is a self-distancing curve from K with starting point $x_0 \in \partial K$, then*

$$\gamma^K \subset cl(\mathbb{R}^2 \setminus (\mathfrak{J}(K, x_0) \cup K)). \quad (52)$$

Proof. Let z be the first crossing point of the left and right involutes of K starting at x_0 . Then

$$Int(\mathfrak{J}(K, x_0)) = \mathfrak{G}(K, z).$$

By contradiction, if γ^K has a point $w \in \mathfrak{G}(K, z)$, then, by Theorem 4.5, the following inclusion holds

$$\gamma_w^K \subset cl(\mathfrak{G}(K, w));$$

since, by the distancing from K property, γ^K has in common with K only the starting point x_0 then, the following inclusion

$$\gamma_w^K \setminus \{x_0\} \subset cl(\mathfrak{G}(K, w)) \setminus \partial K$$

holds too. Moreover by (48) the set $cl(\mathfrak{G}(K, w)) \setminus \partial K$ has positive distance from the $\mathbb{R}^2 \setminus \mathfrak{G}(K, z)$; then $\gamma_w^K \setminus \{x_0\}$ has a positive distance from $\mathbb{R}^2 \setminus \mathfrak{G}(K, z) = \mathbb{R}^2 \setminus Int(\mathfrak{J}(K, x_0))$. This is in contradiction with $x_0 \in \partial \mathfrak{J}(K, x_0)$. \square

Corollary 4.7. *Let γ be a SDC and $z_1 \in \gamma$ then*

$$\gamma \setminus \gamma_{z_1} \subset cl(\mathbb{R}^2 \setminus \mathfrak{J}(co(\gamma_{z_1}), z_1)).$$

Proof. Since $\gamma \setminus \gamma_{z_1}$ is a self-distancing curve from $co(\gamma_{z_1})$ and $z_1 \in \partial co(\gamma_{z_1})$ (see [9, (i) of Lemma 4.6], then Theorem 4.6 applies to $\gamma^K = \gamma \setminus \gamma_{z_1}$ with $K = co(\gamma_{z_1})$. \square

Definition 4.8. *Let γ be a SCD. If $z_1, z \in \gamma$, with $z_1 \preceq z$ let*

$$\gamma_{z_1, z} := \gamma \setminus \gamma_{z_1}.$$

For $z \notin K$, let K^z be the cap body, introduced in (6). Next theorem shows the principal result on bounding regions for arcs of a SDC γ .

Theorem 4.9. *Let K be a convex body and let γ be a SDC_K . If $z_1, z \in \gamma$, with $z_1 \preceq z$ then*

$$\gamma_{z_1, z} \subset cl(\mathfrak{G}(K, z) \setminus \mathfrak{J}(K^{z_1}, z_1)). \quad (53)$$

Proof. First let us notice that $\gamma_{z_1, z}$ has the distancing from K and from the set point $\{z_1\}$ property, thus by Proposition 2.6 it has the distancing from K^{z_1} property. Then the inclusion (53) follows from Theorems 4.5 and 4.6. \square

Let us conclude the section with the following inclusion result for \mathfrak{J} -fences.

Theorem 4.10. *Let K, H be two convex bodies not reduced to a point, $K \subset H$. Let $x_0 \in \partial K \cap \partial H$. Then*

$$\mathfrak{J}(K, x_0) \subset \mathfrak{J}(H, x_0).$$

Proof. The boundary of $\mathfrak{J}(H, x_0)$ consists of two arcs of the left and right involutes of H starting at x_0 . By Corollary 3.13 they are SDC_H , then they are SDC_K ; therefore by Theorem 4.6 they cannot intersect the boundary of $\mathfrak{J}(K, x_0)$. \square

4.1 Minimally connecting plane steepest descent curves

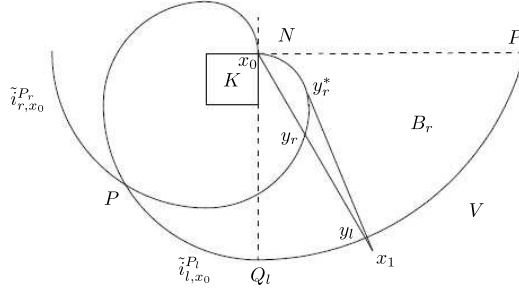


Figure 4: The regions N , B_r , V when K is a square.

Given a point $x_1 \notin K$, the segment joining it with its projection x_0 on ∂K is a SDC_K which *minimally* connects the two points.

This subsection is devoted to consider when it would be possible to connect a given point x_0 on the boundary of a plane convex body K , with an arbitrarily given point $x_1 \notin K$, by using a steepest descent curve $\gamma \in SDC_K$. Let us denote with Γ_{x_0, x_1}^K the class of the curves $\gamma \in SDC_K$ starting at x_0 and ending at x_1 .

Definition 4.11. *Let γ be a SDC with end point y and η be a SDC with starting point y ; let us denote by $\gamma \star \eta$, the curve joining γ with η in the natural order, if it is a SDC curve.*

Theorem 4.12. *Let $x_0 \in \partial K, x_1 \notin K$. Then $\Gamma_{x_0, x_1}^K \neq \emptyset$ iff*

$$x_1 \in cl(\mathbb{R}^2 \setminus (\mathfrak{J}(K, x_0)) \cup K). \quad (54)$$

If (54) holds, there exist at most two $\eta_i \in \Gamma_{x_0, x_1}^K, i = 1, 2$ such that the following properties are true

$$\forall \gamma \in \Gamma_{x_0, x_1}^K \Rightarrow co(\eta_1) \subset co(\gamma) \quad \text{or} \quad co(\eta_2) \subset co(\gamma) \quad (\text{or both}) \quad (55)$$

and

$$\forall \gamma \in \Gamma_{x_0, x_1}^K \Rightarrow |\gamma| \geq \min_{i=1,2} \{|\eta_i|\}. \quad (56)$$

Proof. Let $\gamma \in \Gamma_{x_0, x_1}^K$. From (52) of Theorem 4.6, since $x_1 \in \gamma$, then (54) follows.

Let us prove now that (54) is sufficient. Let us notice that $\mathbb{R}^2 \setminus (\mathfrak{J}(K, x_0) \cup K)$ can be divided in four regions N, B_l, B_r, V , see Fig. 4, defined as follows

- i) the closed normal sector $N := x_0 + N_K(x_0)$ is the angle bounded by the two half lines t_l, t_r tangent at $x_0 \in \partial K$ to the left and right involute $i_l := i_{l, x_0}, i_r := i_{r, x_0}$ respectively; this angle can be reduced to an half line, starting at x_0 ;
- ii) let P be the first crossing point between i_l and i_r , see Theorem 3.14; i_l is a SDC until to $i_l(\vartheta_l^*)$, which will be a point Q_l following P ; after Q_l the involute i_l is no more a SDC, see i) of Theorem 3.12.

Let us change i_l after Q_l with j_{l, Q_l} , the left involute of $co(K \cup i_l^{Q_l})$ at Q_l . Let us define P_l be the first intersection point of j_{l, Q_l} with ∂N , and let

$$\tilde{i}_{l, x_0}^{P_l} := i_{l, x_0}^{Q_l} \star j_{l, Q_l}^{P_l}.$$

It is not difficult to see that $\tilde{i}_{l, x_0}^{P_l} \in \Gamma_{x_0, P_l}^K$. Changing the left with the right, \tilde{i}_{r, x_0} and the point P_r can be constructed. Let B_r the union of the arc $i_{r, x_0}^P \setminus \{P\}$ with the plane open region bounded by the segment $x_0 P_l$, the arc i_{r, x_0}^P and the arc \tilde{i}_l^{P, P_l} ; let B_l the union of the arc $i_{l, x_0}^P \setminus \{P\}$ with the plane open region bounded by the segment $x_0 P_r$, the arc i_{l, x_0}^P and the arc \tilde{i}_r^{P, P_r} ;

- iii) let V the remaining region, i.e. $V = \mathbb{R}^2 \setminus (K \cup \mathfrak{J}(K, x_0) \cup N \cup B_l \cup B_r)$.

Let $x_1 \in B_r \cup V$. On the oriented curve $\tilde{i}_{r, x_0}^{P_r}$ there are two points so that their tangent lines contain x_1 . Let y_r^* be the first tangency point.

Let y_l, y_r be the intersection points of the half line m starting at x_0 and containing x_1 , with $\tilde{i}_{l, x_0}^{P_l}$ and with $\tilde{i}_{r, x_0}^{P_r}$ respectively, see Fig. 4.

Under the assumption (54), x_1 belongs to one of the four regions N, B_l, B_r, V ; let us prove now (55), (56) in the four corresponding cases.

1. If $x_1 \in N$, then let $\eta_1 = \eta_2 \in \Gamma_{x_0, x_1}^K$ be the segment $x_0 x_1$. Then (55), (56) are trivial.
2. Let $x_1 \in V$. Let

$$\eta_r := \tilde{i}_{r, x_0}^{y_r^*} \star y_r^* x_1. \quad (57)$$

The curve η_r is a SDC_K , since the normal lines at all the points on the segment $y_r^* x_1$ have the same directions and support $\tilde{i}_{r, x_0}^{y_r^*}$ up to y_r^* ; then η_r is a SDC_K and joins x_0 with x_1 . Similarly is defined the SDC_K

$$\eta_l = \tilde{i}_{l, x_0}^{y_l^*} \star y_l^* x_1. \quad (58)$$

Thus Γ_{x_0, x_1}^K is not empty and contains at least the two elements η_l, η_r . Let us consider the connected closed curve

$$c_{x_1} := \tilde{i}_{r, x_0}^{y_r} \cup y_r y_l \cup \tilde{i}_{l, x_0}^{y_l}.$$

Let $\gamma \in \Gamma_{x_0, x_1}^K$, let $T \ni t \rightarrow x(t) \in \gamma$ be a continuous parameterization of γ . Let us project from x_0 the curve γ on c_{x_1} and let D be this projection. That is, for $t \in T$, let $\lambda_t := \{x_0 + \lambda x(t), 0 \leq \lambda\}$ and let

$$D = \cup_{t \in T} (c_{x_1} \cap \lambda_t). \quad (59)$$

Clearly D is a closed connected subset of c_{x_1} containing x_0 and the segment $y_r y_l$. Thus D contains at least one of the two connected components of c_{x_1} joining x_0 with the y_r, y_l . Therefore the inclusions

$$(\tilde{i}_{r, x_0}^{y_r} \cup y_r y_l) \subset D, \quad (60)$$

or

$$(\tilde{i}_{l, x_0}^{y_l} \cup y_r y_l) \subset D, \quad (61)$$

(or both) hold.

Assume that (60) holds and let η_r be defined as in (57). Since, by construction of D , the set $co(D \cup \{x_1\})$ is contained in $co(\gamma)$, then

$$co(\eta_r) \subset co(\gamma).$$

Similarly if (61) holds, then

$$co(\eta_l) \subset co(\gamma),$$

with η_l define by (58). Then (55) is proved. It is not difficult to see that the region bounded by $\gamma \cup x_0 x_1$ contains the convex region bounded by $\tilde{i}_{r, x_0}^{y_r^*} \cup y_r^* x_1 \cup x_0 x_1$. Thus the bound

$$|\gamma| \geq |\tilde{i}_{r, x_0}^{y_r^*}| + |y_r^* x_1|$$

holds; similary procedure can be used for left case. This proves (56).

3. Let $x_1 \in B_r$; the same argument as in the case 2. can be carried on up to the inclusions (60), (61). As in the step 2., when case (60) holds, the curve η_r can be constructed and η_r is a SDC_K .

Let us show that if $x_1 \in B_r$ then (61) cannot occur, so the curve η_l can not to be constructed.

Let us argue by contradiction. If (61) occurs, then let $z_1 \neq x_0$ be the first point where γ crosses the half line x_0P . The point z_1 exists, since under the assumption (61), $P \in D$. Then (Theorem 4.6) z_1 does not belong to the open segment x_0P . Then, from (61) (see Definition 3.17)

$$co(\gamma_{z_1}) \supset \mathfrak{J}_l(K, x_0). \quad (62)$$

Moreover $\gamma \setminus \gamma_{z_1} \cup \{z_1\}$ is a $SDC_{co(K \cup \gamma_{z_1})}$, see Remark 2.5. Then by (62) it is a $SDC_{co(K \cup \mathfrak{J}_l(K, x_0))}$. Let us consider the convex body $H = co(K \cup i_{l, x_0}^P)$. Since

$$co(K \cup \gamma_{z_1}) \supset co(K \cup \mathfrak{J}_l(K, x_0)) \supset H,$$

then $\gamma \setminus \gamma_{z_1}$ is a deleted SDC_H . From Theorem 4.9, with H in place of K , x_1 in place of z , it follows that

$$\gamma_{z_1, x_1} \subset cl(\mathbb{R}^2 \setminus \mathfrak{J}(H^{z_1}, z_1)).$$

Let P_1 where the right tangent from z_1 to H crosses the arc $\tilde{i}_{l, P}^{P_1}$. Let us notice that $\tilde{i}_{l, P}^{P_1}$ is also an arc of the left involute of H at P . Moreover $\tilde{i}_{l, P_1}^{P_1, P_1}$ is an arc of the left involute of H^{z_1} starting at P_1 , then it is parallel to the left involute of H^{z_1} at z_1 until it crosses the sector N ; it turns out that

$$B_r \subset \mathfrak{J}(H^{z_1}, z_1).$$

From the two previous inclusions a contradiction comes out since

$$x_1 \in \gamma_{z_1, x_1} \cap B_r = \emptyset.$$

Then (61) cannot occur.

4. The case $x_1 \in B_l$ is similar to the previous one.

The proof is complete. \square

Definition 4.13. Under the assumptions of Theorem 4.12, let us define E_{x_0, x_1}^K the set of $\eta_i, i = 1, 2$ (possibly coinciding) as they are constructed in the proof of Theorem 4.12, which satisfy (55), (56). These curves will be called minimally connecting steepest descent curves for the class Γ_{x_0, x_1}^K .

Definition 4.14. Let $\gamma : T \ni t \rightarrow x(t)$ be an absolutely continuous curve and let $x(t)$ be a point of γ , with tangent vector $\dot{x}(t)$. Let

$$\mathcal{H}_{x(t)} := \{y \in \mathbb{R}^2 : \langle \dot{x}(t), y - x(t) \rangle \leq 0\}.$$

$\mathcal{H}_{x(t)}$ is an half plane bounded by the normal line to γ at $x(t)$ and it is defined almost everywhere in T . For the curve $\gamma \setminus \gamma_{x_1}$ (consisting of the points of γ following x_1) let us define the region:

$$\mathfrak{H}(\gamma, x_1) := \bigcap_{x_1 \preceq x(t), x(t) \in \gamma} \mathcal{H}_{x(t)}.$$

If $\mathfrak{H}(\gamma, x_1) \neq \emptyset$, then it is a convex set.

If γ is a SDC (γ is a SDC_K), then condition (1) (respectively (4)) implies that

$$\gamma_{x_1} \subset \mathfrak{H}(\gamma, x_1) \quad (\text{respectively} \quad \gamma_{x_1} \cup K \subset \mathfrak{H}(\gamma, x_1)). \quad (63)$$

Theorem 4.15. Let $x_0 \in \partial K, x_1 \notin K$. Let γ_1 be a SDC with first point x_1 . Necessary and sufficient conditions for the existence of a curve γ , self-distancing curve from K , starting at x_0 and satisfying $(\gamma \setminus \gamma_{x_1}) \cup \{x_1\} = \gamma_1$, are as follows:

- (a) $x_1 \in cl(\mathbb{R}^2 \setminus \mathfrak{J}(K, x_0))$;
- (b) there exists $\eta \in E_{x_0, x_1}^K$ such that $(K \cup \eta) \subset \mathfrak{H}(\gamma_1, x_1)$;

moreover if (a) and (b) are satisfied, then $\gamma_1 \in SDC_{co(K \cup \eta)}$.

Proof. (a) is necessary by Theorem 4.6. (b) is necessary by Theorem 4.12 and by (63) since $\mathfrak{H}(\gamma_1, x_1) = \mathfrak{H}(\gamma, x_1)$. Vice versa if (a), (b) hold, let us define $\gamma := \eta \star \gamma_1$; then (by definition of E_{x_0, x_1}^K) η is a SDC_K ; thus, γ is a SDC_K too and γ_1 is a $SDC_{co(K \cup \eta)}$ (see Remark 2.5). \square

5 Self-distancing sets and steepest descent curves

A self-distancing set σ will be called SDC-extendible if there exists a steepest descent curve γ such that $\sigma \subset \gamma$.

This section is devoted to investigate the following question:

Can a self-distancing set σ be extended to a steepest descent curve γ ?

Let us call Γ_σ the family of SDC γ wick extends σ . The following example shows that Γ_σ can be empty.

Example 5.1. Let us consider in a coordinate system xy the points:

$$x_1 = (0, 0), x_2 = (0, 2), x_3 = (1, \sqrt{8}), x_4 = (-1, \sqrt{8}).$$

The set $\tilde{\sigma} = \{x_i, i = 1, \dots, 4\}$ is a self-distancing set not SDC-extendibile.

Proof. By contradiction let $\gamma \in \Gamma_{\tilde{\sigma}}$, then any point x on the arc γ_{x_3, x_4} satisfies the inequalities

$$3 = |x_3 - x_1| \leq |x - x_1| \leq |x_4 - x_1| = 3, \quad |x_3 - x_2| \leq |x - x_2|.$$

That is $x \in \partial B(x_1, 3)$ and $x \in \mathbb{R}^2 \setminus B(x_2, |x_3 - x_2|)$. Since $\partial B(x_1, 3) \cap (\mathbb{R}^2 \setminus B(x_2, |x_3 - x_2|)) = \{x_3, x_4\}$, the arc γ_{x_3, x_4} consists of two points only, that is impossible. \square

Next theorem gives a necessary condition (64) in order to extend a finite self-distancing set σ to a SDC; this condition is based on the bounding sets introduced in §3.1.

Let us define σ_x as the subset of σ consisting of the point x and of the previous ones on σ (consistently with (2)).

Theorem 5.2. *Let σ be a self expanding SDC-extendible set, then for all $x_0 \in \sigma$ such that $\sigma_{x_0} \neq \{x_0\}$, the following inclusion*

$$(\sigma \setminus \sigma_{x_0}) \subset cl(\mathbb{R}^2 \setminus \mathfrak{J}(co(\sigma_{x_0}), x_0)) \quad (64)$$

holds.

Proof. Let $\gamma \in \Gamma_{\sigma}$, then $\sigma \subset \gamma$ and γ is a SDC. Then $co(\gamma_{x_0}) \supset co(\sigma_{x_0})$, $x_0 \in \partial co(\gamma_{x_0}) \cap \partial co(\sigma_{x_0})$ (see [9, (i) of Lemma 4.6]); from Theorem 4.10 ,

$$\mathfrak{J}(co(\gamma_{x_0}), x_0) \supset \mathfrak{J}(co(\sigma_{x_0}), x_0);$$

moreover $\sigma \setminus \sigma_{x_0} \subset \gamma \setminus \gamma_{x_0}$ and from Corollary 4.7,

$$\gamma \setminus \gamma_{x_0} \subset cl(\mathbb{R}^2 \setminus \mathfrak{J}(co(\gamma_{x_0}), x_0)).$$

The previous inclusions prove (64). \square

Remark 5.3. *In the Example 5.1 it has been proved, in a simple way, that $\tilde{\sigma}$ is not SDC-extendible. Another way to prove this fact is to check that the condition (64) does not hold for the point x_4 ; let us notice that $\partial \mathfrak{J}(co(\tilde{\sigma}_{x_3}), x_3) \cap \{x \leq 0, y \geq 0\}$ consists of a circular arc centered at x_1 with radius $2 + \sqrt{13 - 4\sqrt{8}}$; then it is easy to see that x_4 is in the interior of $\mathfrak{J}(co(\tilde{\sigma}_{x_3}), x_3)$ and (64) is not satisfied.*

Let us show in the following example that (64) is not sufficient for a self-distancing set σ to be SDC-extendible.

Example 5.4. *Let us consider in a coordinate system xy the points:*

$$\xi_1 = (0, 0), \xi_2 = (0, 2), \xi_3 = (2, 0), \xi_4 = (\rho, 2).$$

For $\sqrt{8} < \rho < \pi$, the set $\sigma := \{\xi_i, i = 1, \dots, 4\}$ is a self-distancing set satisfying the condition (64) not SDC-extendible.

Proof. It is easy to see that σ is a self-distancing set. Moreover the initial piece of the left involute of $co(\{\xi_1, \xi_2, \xi_3\})$ starting at ξ_3 consists of a circular arc centered at ξ_2 of ray $\sqrt{8}$ and amplitude $\frac{3}{4}\pi$. Then $\xi_4 \notin \mathfrak{J}(co(\{\xi_1, \xi_2, \xi_3\}), \xi_3)$ and (64) is verified with $x_0 = \xi_3$, $\sigma \setminus \sigma_{x_0} = \{\xi_4\}$. Trivially (64) is verified also at $x_0 = \xi_2$. Let us prove now that Γ_σ is empty. By contradiction let $\gamma \in \Gamma_\sigma$. Let us consider γ_{ξ_3} . Since ξ_2, ξ_3 have the same distance from ξ_1 , arguing as in Example 5.1, γ_{ξ_3} is a circular arc C centered at ξ_1 from ξ_2 to ξ_3 . Since the arc γ_{ξ_3} has the distancing property from the segment $\xi_1\xi_2$, it is necessarily the arc of amplitude $\pi/4$ and not the complementary arc. Let $\eta = \xi_1\xi_2 \star C$. Since $C \subset \gamma$ and $\xi_2, \xi_3 \in \gamma$, then $co(\eta) \subset co(\gamma_{\xi_3})$. Thus by Corollary 4.10

$$\mathfrak{J}(co(\gamma_{\xi_3}), \xi_3) \supset \mathfrak{J}(co(\eta), \xi_3).$$

Since the segment $\xi_2\xi_4$ is tangent to η at ξ_2 and it has length ρ , less than π , the length of the arc η_{ξ_2, ξ_3} , then

$$\xi_4 \in Int(\mathfrak{J}(co(\gamma_{\xi_3}), \xi_3)).$$

This is in contradiction with Corollary 4.7 at the point ξ_3 . \square

Let us introduce definitions and preliminary facts needed to obtain necessary and sufficient conditions for the extendibility of a self-distancing set σ structured as follows.

Definition 5.5. Let us denote with $\tilde{\cup}_i \sigma_i$ a self-distancing set with a finite (or countable) family of closed connected components $\sigma_1, \sigma_2, \dots, \sigma_n, \dots$, ordered as the points of σ , that is if $i < j, x \in \sigma_i, y \in \sigma_j \Rightarrow x \preceq y$. Let x_i^- be the first point and let x_i^+ be the last point of σ_i ; if they are distinct (that is σ_i does not reduce to a point) as noticed in the introduction ([3, Theorem 3.3] and [9, Theorem 4.10]), σ_i is a SDC and it will be denoted by γ_i .

Lemma 5.6. Let $\sigma = \tilde{\cup}_i \sigma_i$ be a self-distancing set. A necessary condition for $\Gamma_\sigma \neq \emptyset$ is that for all components σ_i , which are curves γ_i , the following fact

$$\cup_{j=1}^i \sigma_j \subset \mathfrak{H}(\gamma_{i+1}, x_{i+1}^-), \quad (65)$$

holds.

Proof. Let $\gamma \in \Gamma_\sigma$. Then $\cup_{j=1}^i \sigma_j \subset \gamma_{x_{i+1}^-}$. Then (65) follows from (63). \square

Definition 5.7. Let $\sigma = \tilde{\cup}_i \sigma_i$ be a self-distancing set. A subfamily $E_\sigma \subset \Gamma_\sigma$ is called essential for Γ_σ if the following facts

$$a) \quad \forall \gamma \in \Gamma_\sigma \exists \eta \in E_\sigma : co(\eta) \subset co(\gamma), \quad (66)$$

$$b) \quad \gamma \in \Gamma_\sigma \Rightarrow |\gamma| \geq \min\{|\eta|, \eta \in E_\sigma\}, \quad (67)$$

$$c) \quad \gamma \in \Gamma_\sigma, |\gamma| = \min\{|\eta|, \eta \in E_\sigma\} \Rightarrow \gamma \in E_\sigma \quad (68)$$

hold.

It can happen that $E_\sigma = \emptyset$. If $\Gamma_\sigma = \emptyset$, let us define $E_\sigma = \emptyset$ essential for Γ_σ .

Let us start to study a self-distancing set with two closed connected components.

Lemma 5.8. *Let $\sigma^* = \tilde{\cup}_{i=1,2} \sigma_i$ be a self-distancing set and let ρ be the segment joining x_1^+, x_2^- . There are five possibilities:*

p1) *Let $\sigma_1 = \{x_1\}, \sigma_2 = \{x_2\}$, then $E_{\sigma^*} = \{\rho\} \neq \emptyset$ is essential for Γ_{σ^*} .*

p2) *Let $\sigma_1 = \{x_1\}, \sigma_2 = \gamma_2$ (γ_2 is a SDC); then a necessary and sufficient condition for the extensibility of σ^* is*

$$\sigma_1 \subset \mathfrak{H}(\gamma_2, x_2^-); \quad (69)$$

moreover $E_{\sigma^*} = \{\rho \star \gamma_2\}$ is essential for Γ_{σ^*} .

p3) *Let $\sigma_1 = \gamma_1$ be a SDC, $\sigma_2 = \{x_2\}$; then a necessary and sufficient condition for the extensibility of σ^* is*

$$\sigma_2 \subset cl(\mathbb{R}^2 \setminus (\mathfrak{J}(co(\gamma_1), x_1^+) \cup co(\gamma_1))); \quad (70)$$

moreover $E_{\sigma^*} = \{\gamma_1 \star \eta : \eta \in E_{x_1^+, x_2^-}^{co(\gamma_1)}\}$ (see Definition 4.13).

p4) *Let $\sigma_1 = \gamma_1, \sigma_2 = \gamma_2$; then a necessary and sufficient condition for the extensibility of σ^* is that there exists a SDC η such that:*

$$\eta \in E_{x_1^+, x_2^-}^{co(\gamma_1)} \quad \text{and} \quad \sigma_1 \cup \eta \subset \mathfrak{H}(\gamma_2, x_2^-); \quad (71)$$

the related essential family is $E_{\sigma^*} = \{\gamma_1 \star \eta \star \gamma_2 : \eta \text{ satisfies (71)}\}$.

p5) *If in the cases p2), p3), p4) the corresponding necessary and sufficient conditions do not hold, then*

$$E_{\sigma^*} = \Gamma_{\sigma^*} = \emptyset.$$

Proof. case p1) is trivial; in the case p2) the inclusion (69) follows from Lemma 5.6 with $i = 2$ in (65). It is also trivial that it is sufficient. The case p3) follows from Theorem 4.12 with $K = co(\gamma_1)$. The case p4) follows from Theorem 4.15 with γ_2 in place of γ_1 , $K = co(\gamma_1)$, x_1^+ in place of x_0 , x_2^- in place of x_1 . \square

An easy sufficient condition to check if $\sigma = \tilde{\cup}_i \sigma_i$ is extendible is the following

Theorem 5.9. *Let $\sigma = \tilde{\cup}_i \sigma_i$ be a self-distancing set. Let $\sigma^{(i)} = \tilde{\cup}_{j=1}^i \sigma_j$. If (65) and*

$$x_{i+1}^- \in N_{co(\sigma^{(i)})}(x_i^+), \quad \forall i \geq 1 \quad (72)$$

hold, then $\Gamma_\sigma \neq \emptyset$ and $\overline{\gamma}$, which linearly and orderly connects σ_i, σ_{i+1} with the segments $x_i^+ x_{i+1}^-$, is a SDC and it has minimal length in Γ_σ .

Proof. Let us argue by induction on the self-distancing set $\sigma^{(i)}$. The case $i = 1$ is contained in Lemma 5.8, since the assumptions (65) and (72) are enough to get the corresponding assumptions in the cases p1),p2),p3),p4). Moreover in the case p4) the curve $\eta = x_1^+ x_2^-$ is such that $\gamma^{(2)} = \gamma_1 \star \eta \star \gamma_2$ is the SDC of minimal length extending $\sigma^{(2)}$.

Let $\gamma^{(i)}$ the curve of minimal length extending $\sigma^{(i)}$. Since the normal sector to $co(\sigma^{(i)})$ at x_i^+ coincides with the sector N in the proof of Theorem 4.12, with x_i^+ in place of x_0 , x_{i+1}^- in place of x_1 then, the assumption (72) implies that the case 1. of the proof of Theorem 4.12 occurs. It follows that

$$\gamma^{(i+1)} = \gamma^{(i)} \star x_i^+ x_{i+1}^- \star \sigma_{i+1}$$

is of minimal length in $\Gamma_{\sigma^{(i+1)}}$. Then, $\overline{\gamma} = \cup_i \gamma^{(i+1)}$ is of minimal length in Γ_σ . \square

Remark 5.10. *Since, as noticed in [12, II, Section 2], $\forall u, w \in \sigma_{x_i^+}$ the angle $ux_i^+ w$ has opening less than $\pi/2$, thus the related normal sector in (72) has opening greater or equal than $\pi/2$; then to check that x_{i+1}^- satisfies (72), is easier than to check that x_{i+1}^- is outside of the \mathfrak{F} -fence as in (70).*

Lemma 5.6 and Theorem 5.9 give only necessary and only sufficient conditions, respectively, for the extensibility of self expanding sets. Let us give definitions in order to get necessary and sufficient conditions.

Definition 5.11. *Let $\sigma = \tilde{\cup}_j \sigma_j$ be a self-distancing set. Let $E_i, i = 2, \dots, n, \dots$ be defined by induction as follows:*

E_2 is the essential family related to $\tilde{\cup}_{j=1}^2 \sigma_j$, as given by Lemma 5.8; if $i \geq 2$, the E_{i+1} related to $\tilde{\cup}_{j=1}^{i+1} \sigma_j$ is defined as follows:

i) if $E_i = \emptyset$ then $E_{i+1} = \emptyset$;

ii) if $E_i \neq \emptyset$, let us consider for all $\eta \in E_i$ the essential family $E(\eta)$ (see Lemma 5.8) related to $\eta \tilde{\cup} \sigma_{i+i}$ (see Definition 5.5). Let $E_{i+1} = \cup_{\eta \in E_i} E(\eta)$.

Let us notice that $\{E_i\}$ is ordered by inclusion and E_{i+1} , if it is not empty, consists of 2^i curves at most.

Theorem 5.12. *Let $\sigma = \tilde{\cup}\sigma_j$ be a self expanding set and let $E_2, E_3, \dots, E_i, \dots$ be the sequence (finite or countable) of the essential families associated to σ . Then $\Gamma_\sigma \neq \emptyset$ iff $\forall i \geq 2$ the essential family E_i is not empty.*

Proof. If there exists $\gamma \in \Gamma_\sigma$ then, for all $i \geq 1$, $\gamma_{x_{i+1}^+} \in \Gamma_{\tilde{\cup}_{j=1}^{i+1}\sigma_j}$; thus $E_{i+1} \neq \emptyset$ by Theorem 4.12. Vice versa if at each step $i \geq 1$ the essential family $E_{i+1} \neq \emptyset$, then, by definition, there exists a sequence $\{\eta^{i+1}\}$ of SDC such that $\eta^{i+1} \in E_{i+1}$ and such that $\eta^s \subset \eta^{s+1}$, $s \geq 1$ (that is, at each step, η^{s+1} is an extension of a previous one η^s) and $\eta^{s+1} \in E_{\tilde{\cup}_{j=1}^{s+1}\sigma_j}$, see Definition 5.11. Then is well defined $\gamma = \cup_{i=1}^\infty \eta^{i+1}$; obviously $\gamma \in \Gamma_\sigma$. \square

Open problem: In the present work only two dimensional problems are studied. In three (or more) dimensions the construction of boundary regions to a SDC and to a SDC_K is open. The boundary regions should probably be constructed by using the space involutes of the geodesics curves on ∂K .

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